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THE ISOPERIMETRICAL PROBLEM ON ANY SURFACE.

By J. K. Whittemore.

One of the most famous problems of mathematics is the so-called isoperimetrical problem in the plane: to find among closed curves of given perimeter that enclosing the greatest area. That the solution of this problem is the circle was known to Pappus* who wrote about 300 A. D.

The problem can be solved by the methods of the Calculus of Variations †

I give here a brief solution of the problem stated in a slightly different form. Let it be required to draw a curve of given length, L, between two given points of the X-axis $(a, 0)$, and $(b, 0)$, supposing

$$0 < b - a < L < \frac{\pi}{2}(b - a),$$

such that the area between the curve and the X-axis shall be a maximum. Analytically, this is to make the integral I a maximum, subject to the condition that the integral L shall have a given value, where

$$I = \int_a^b y\, dx, \qquad L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

The method of the Calculus of Variations consists in forming the integral,

$$U = \int_a^b (y + \lambda \sqrt{1 + y'^2})\, dx,$$

where λ is an undetermined constant, and then requiring that the first variation of U shall vanish. The differential equation that thus results is the following:

$$1 - \lambda \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = 0. \tag{1}$$

The expression $\frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}}$ is the curvature of the curve $y = y(x)$.

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* William Thomson, *Popular Lectures and Addresses*, vol. 2, p. 578.

† See *e. g.* Jordan, *Cours d'analyse*, 2d ed., vol. 3, p. 497, where necessary conditions are obtained; Kneser, *Lehrbuch der Variationsrechnung*, pp. 123 and 136, where a complete solution is given.

Hence (1) tells us that the radius of curvature of the curve sought is constant and equal to λ. The solution is then a circle of radius λ, which may be determined so as to make the length of the arc, between the two points, equal to L.

A natural generalization is the same problem on any surface. This I shall now consider, proving that the curve sought is, at all points at which it has a tangent that turns continuously, of *constant geodesic curvature.* The necessary condition has long since been obtained; *cf.* for example, Knoblauch, *Krumme Flächen*, p. 251; Kneser, *l. c.*, p. 125, and Darboux, *Théorie des surfaces*, vol. 3, p. 151. It is my purpose here to give a treatment different from those which I have seen, which from the point of view of differential geometry is particularly simple.

We suppose given a surface and on it a curve connecting two of its points. Assume an arbitrary system of curvilinear coordinates (u, v) on the surface, and let the coordinates of the given points be (u_0, v_0) and (u_1, v_1) respectively; the equation of the given curve,

$$v = f(u).$$

The problem before us is *to find a curve, $v = \phi(u)$, joining the two given points (u_0, v_0) and (u_1, v_1), having a given length L, and such that the area of the portion of the surface between the two curves, $v = f(u)$ and $v = \phi(u)$, shall be a maximum.*

The geodesic curvature $1/\rho_\psi$ of a curve on the surface, $\psi(u, v) = 0$, is given by the following equation :*

$$\frac{1}{\rho_\psi} =$$

$$\frac{1}{\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\frac{F\frac{\partial\psi}{\partial v}-G\frac{\partial\psi}{\partial u}}{\sqrt{E(\frac{\partial\psi}{\partial v})^2-2F\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v}+G(\frac{\partial\psi}{\partial u})^2}}+\frac{\partial}{\partial v}\frac{F\frac{\partial\psi}{\partial u}-E\frac{\partial\psi}{\partial v}}{\sqrt{E(\frac{\partial\psi}{\partial v})^2-2F\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v}+G(\frac{\partial\psi}{\partial u})^2}}\right\}$$

If for $\psi(u,v)$, we write $v - \phi(u)$, we have for the geodesic curvature

$$\frac{1}{\rho_g} = \frac{1}{\sqrt{EG - F^2}}\left\{\frac{\partial}{\partial u}\frac{F + G\phi'}{\sqrt{E + 2F\phi' + G\phi'^2}} - \frac{\partial}{\partial v}\frac{F\phi' + E}{\sqrt{E + 2F\phi' + G\phi'^2}}\right\}$$

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* See Bianchi, *Differentialgeometrie*, p. 149.

In these equations E, F, and G are the coefficients of the linear element of the surface; that is to say for any curve drawn on the surface, we have

$$ds^2 = dx^2 + dy^2 + dz^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

The given length L of the curve $v = \phi(u)$ joining the two given points is given by the integral,

$$L = \int_{u_0}^{u_1} \sqrt{E + 2F\phi' + G\phi'^2}\,du$$

The area I which we wish to make a maximum is given by the double integral,

$$I = \int_{u_0}^{u_1}\int_{f(u)}^{\phi(u)} \sqrt{EG - F^2}\,dv\,du.$$

The unknown function $v = \phi(u)$ will be found by causing to vanish the first variation of the integral

$$U = \int_{u_0}^{u_1} \left\{ \int_{f(u)}^{v} \sqrt{EG - F^2}\,dv + \lambda\sqrt{E + 2Fv' + Gv'^2} \right\} du,$$

where λ is a constant.

The differential equation for $v = \phi(u)$ deduced from this condition in the Calculus of Variations is the following:

$$\sqrt{EG - F^2} + \lambda\frac{\dfrac{\partial E}{\partial v} + 2\dfrac{\partial F}{\partial v}v' + \dfrac{\partial G}{\partial v}v'^2}{2\sqrt{E + 2Fv' + Gv'^2}} - \lambda\frac{d}{du}\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} = 0. \quad (2)$$

From this equation we show that the curve $v = \phi(u)$ is of constant geodesic curvature. It is the simplification of this proof by means of a suitably chosen system of curvilinear coordinates that is the main point of this paper.

Consider the differential equation

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0.$$

This equation, being of the first order and of the second degree, may be replaced by two equations each of the first order and of the first degree. Let the general solutions of these two equations be

$$\alpha(u, v) = c, \qquad \beta(u, v) = c'.$$

The curves represented by these equations are the "minimal curves" of the surface. If we choose them as coordinate curves we shall have

$$ds^2 = 2\mathfrak{J}\,d\alpha\,d\beta.$$

Suppose by this change of coordinates the equation of the curve $v = \phi(u)$ becomes $\beta = \theta(\alpha)$. Then for the geodesic curvature we have

$$\frac{1}{\rho_g} = \frac{1}{i\mathfrak{J}\sqrt{2}}\left(\frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{J}}{\theta'}} - \frac{\partial}{\partial\beta}\sqrt{\mathfrak{J}\theta'}\right) \tag{3}$$

The differential equation (2) is of the same general form for any system of curvilinear coordinates. If we use α and β for curvilinear coordinates, it becomes:

$$i\mathfrak{J} + \lambda\sqrt{2\beta'}\,\frac{\partial\sqrt{\mathfrak{J}}}{\partial\beta} - \frac{\lambda}{\sqrt{2}}\frac{d}{d\alpha}\sqrt{\frac{\mathfrak{J}}{\beta'}} = 0. \tag{2'}$$

Here we may write

$$\frac{d}{d\alpha}\sqrt{\frac{\mathfrak{J}}{\beta'}} = \frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{J}}{\theta'(\alpha)}} + \theta'\frac{\partial}{\partial\beta}\sqrt{\frac{\mathfrak{J}}{\theta'(\alpha)}}$$

and thus (2′) assumes the form:

$$i\mathfrak{J} + \frac{\lambda}{\sqrt{2}}\frac{\partial}{\partial\beta}\sqrt{\mathfrak{J}\theta'} - \frac{\lambda}{\sqrt{2}}\frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{J}}{\theta'}} = 0,$$

where θ' is regarded, in the partial differentiations, as a function of α. From this equation it follows that

$$\frac{1}{\lambda} = \frac{1}{i\mathfrak{J}\sqrt{2}}\left(\frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{J}}{\theta'}} - \frac{\partial}{\partial\beta}\sqrt{\mathfrak{J}\theta'}\right),$$

and hence by the aid of (3) we infer that the curve $\beta = \theta(\alpha)$ has the property that

$$\rho_g = \lambda = \text{const.}$$

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